# XOR Local Search for Boolean Brent Equations

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- Random k-SAT and satisfiable, hard-combinatorial problems.





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Random k-SAT and satisfiable, hard-combinatorial problems.

- While CDCL is the dominating SAT solving paradigm, there are problems on which Stochastic Local Search performs significantly better.
  - The largest satisfiable instance of the Boolean Pythagorean Triples problem can be solved using DDFW [Divide and Distribute Fixed Weights] local search in ~one CPU minute. Other algorithms time out.
  - SLS solvers perform well in the search for new matrix multiplication schemes expressed as a SAT problem via the Boolean Brent equations.
- Can we further improve the performance of LS on a class of problems where it already performs best? We look at problems involving XOR constraints, of which matrix multiplication is one instance.

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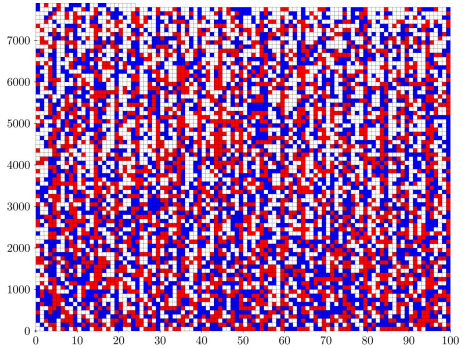


(a) DDFW on Boolean Pythagorean Triples





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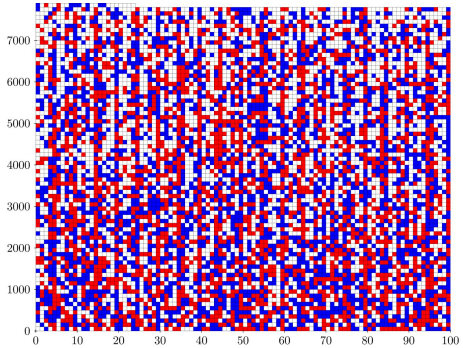


(a) DDFW on Boolean Pythagorean Triples

$$\sum_{\iota=1}^{23} \alpha^{(\iota)}_{i_1,i_2} \beta^{(\iota)}_{j_1,j_2} \gamma^{(\iota)}_{k_1,k_2} = \delta_{i_2,j_1} \delta_{i_1,k_1} \delta_{j_2,k_2}$$

(b) Matrix Multiplication





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# Solving XOR in CNF form

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- To solve problems involving XOR constraints, we have to pick an encoding into CNF.
- Most straightforwardly, we can use a direct encoding – `XOR_d`. But this produces exponentially many clauses.
- The usual linear approach is Tseitin encoding. It recursively breaks off fixed-size chunks and encodes them directly.
- But we pay for linearity. Tseitin encoding introduces auxiliary variables ($y$, underlined). These interact poorly with the SLS algorithm.

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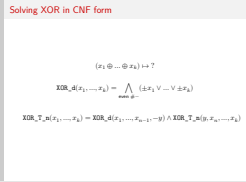


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# Flipping Tseitin

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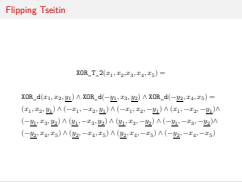


- At its core, SLS proceeds by flipping variables in falsified clauses according to a probabilistic heuristic.
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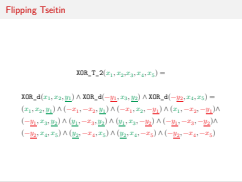


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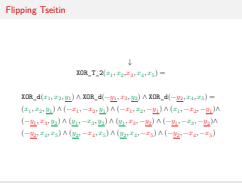


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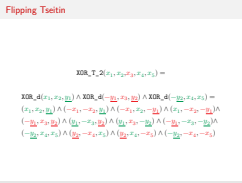


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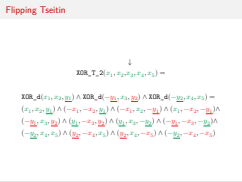


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- We propose to experiment with native XOR representations more widely. In the spirit of DIMACS CNF, an *XNF* format could be used.
- Worth noting that we later found out XNF is already implemented in CryptoMiniSAT.

# xnfSAT: Stochastic Local Search with native XOR

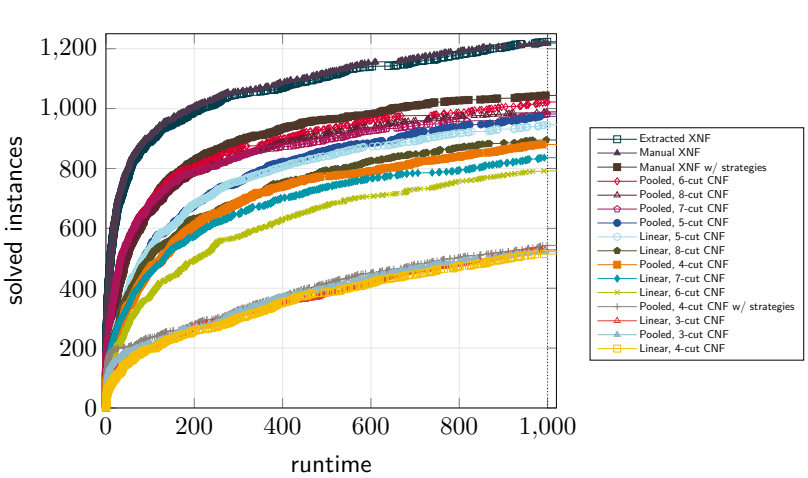
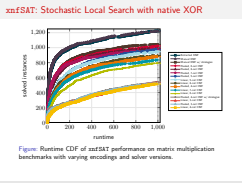



Figure: Runtime CDF of xnfSAT performance on matrix multiplication benchmarks with varying encodings and solver versions.







- To solve XNF, we present xnfSAT, an SLS solver supporting native XOR.
- Its performance on matrix multiplication benchmarks significantly improves upon the best CNF-based solver, YalSAT.
- To go with xnfSAT, we implemented a tool to extract XOR gates from CNF files.

# We build on YalSAT

**Algorithm** YalSAT, a WalkSAT-based solver

1: **for** clause in input file **do**
2: parse and store clause to data structure
3: **end for**
4: preprocess formula
5: $\alpha \leftarrow$ complete initial assignment of truth values
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- `xnfSAT` is based on `YalSAT`. Instructive to understand its outline.
- Unsurprisingly, supporting XOR needs no modifications to the high-level structure.
- We adapt parsing (XNF), preprocessing and variable selection (`pickVarIn`).

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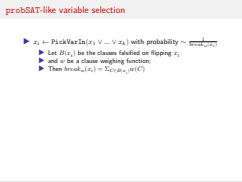


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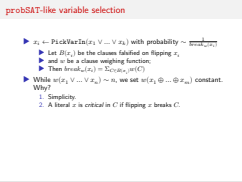


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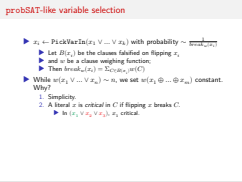


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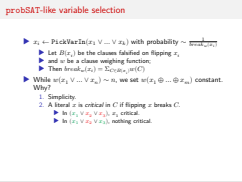


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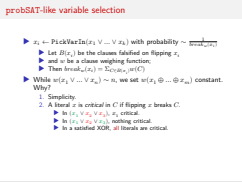


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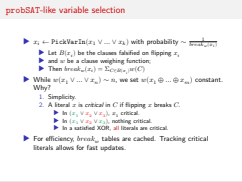
- For efficiency, $break_w$ tables are cached. Tracking critical literals allows for fast updates.





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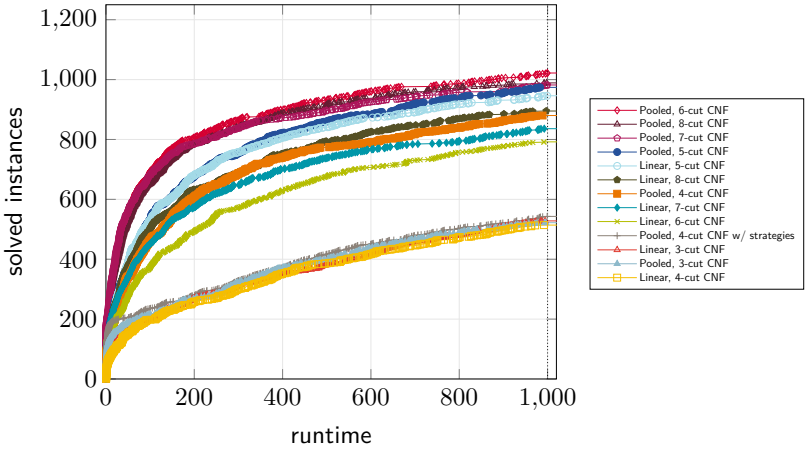


Figure: Runtime CDF of `YalSAT` performance on CNF-encoded matrix multiplication benchmarks.





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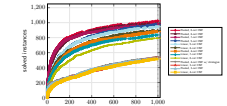

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- On these instances, pooled encodings are better across the board.
- Interestingly, performance initially increases with cutting number and plateaus at 6.

# Not as fast as XNF!

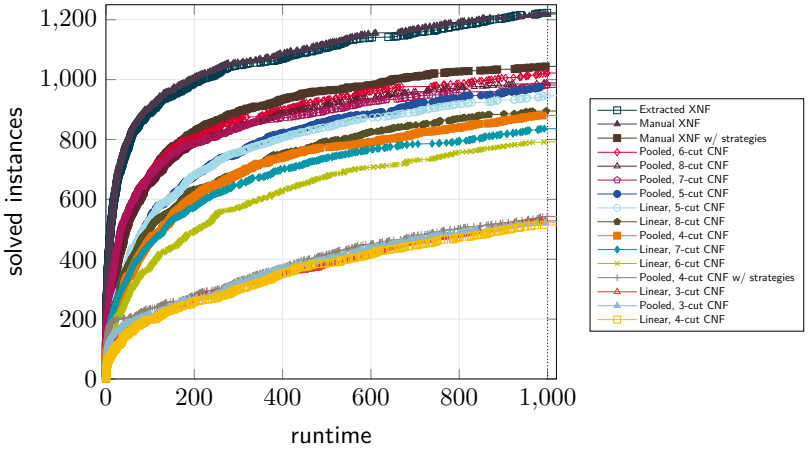
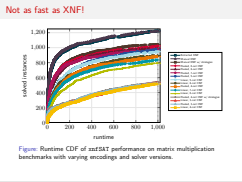



Figure: Runtime CDF of `xnfSAT` performance on matrix multiplication benchmarks with varying encodings and solver versions.








- Here, XNF solving improves over even highly tuned CNF, without having to spend any computational power on optimising the XOR encoding.
- Within a 1000s timeout, our solver operating on XNF can find between 200 and 700 more solutions compared to CNF-based runs in various configurations.

# Conclusion

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- Observed strong performance improvements on matrix multiplication benchmarks where SLS already outperformed CDCL.
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